

A non-linear model of trading mechanism on a financial market

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Abstract

We introduce a prototype model in an attempt to capture some aspects of market dynamics simulating a trading mechanism. The model description starts with a discrete-space, continuous-time Markov process describing arrival and movement of orders with different prices. We then perform a re-scaling procedure leading to a deterministic dynamical system controlled by non-linear ordinary differential equations (ODEs). This allows us to introduce approximations for the equilibrium distribution of the model represented by fixed points of deterministic dynamics.

1 Introduction

This paper proposes a model that takes into account, in a rather stylized form, some aspects of automated trading mechanisms adopted in modern financial markets, in particular, the dynamics of the limit order book.

In short, a limit order book keeps records of arrivals, movements and departures of market participants (traders) who declare their trading positions. An arriving trader may wish to buy or sell at a certain price, and can move his declared price when time progresses. If the declared price is met by a trader with the opposite intention, a trade is recorded: this may lead to disappearance of one or both participants from the market, due to exhaustion of their offers. For a detailed description of some common limit order book models and their applications, see [2, 3, 4] and references therein.

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In the current paper we present a somewhat different model, including elements of queueing behavior of arrived offers; this model is studied by using techniques of asymptotic analysis. An earlier account of this work (in its preliminary version) can be found in [1]. Compared with [1], in the present text we adopt a continuous-time setting for the basic Markov process: it clarifies the meaning of the main parameters of the model and shortens the proof of some of our main results.

The model under consideration is a prototype; at this stage it does not aim to take into account all possible aspects that can be viewed as defining, either theoretically or practically. Instead, we opted for a simplified description which leads to some straightforward, yet instructive, answers.

Our model differs from known models of the limit order book in a number of aspects. Arguably, it can be a subject of criticism (as a number of other proposed models). In particular, the strategic behavior of the model in its current form (and some further details) do not quite match existing mechanisms governing electronic trading on financial markets. Nevertheless, the model shows a certain amount of flexibility, and covers a broad range of situations. Its mathematical advantage is that in the scaling limit under consideration, it leads to a single fixed point.

Our scaling limit is based on suppositions that (i) the number of market participants is large, (ii) during a very short time period only part of them makes a decision of performing a trade or making a move along the price range, and (iii) the probability for any given participant to make such decision is small. This makes it natural to change, in a suitable manner, parameters of original Markov process.

After rescaling, a limiting dynamical system emerges, with a deterministic behavior described by a system of non-linear ordinary differential equations. The rescaling techniques greatly simplify the structure of the model, and this phenomenon extends far beyond basic examples like the current prototype model. As we mentioned earlier, the present paper focuses on a simplified model, with ‘minimal’ number of constant parameters, where some of technically involved issues are absent.

A similar approach is commonly used in the literature on stochastic communication networks; see, e.g., [5, 6, 7, 8] and [9]. We also find similarities, as well as differences, with models proposed (in a different context) in a recent paper [10]; analogies with [10] could be useful for the aforementioned purpose of defining the prices that are appropriate for trades.

In the next section we describe the underlying Markov process. In Section 3 the rescaling of the process is presented and the main results are stated and the proofs are given. The last section contains concluding discussion of various aspects of the model.

2 The underlying Markov process

The rationale for the models below is as follows. We consider a single-commodity market where prices may be at one of N distinct levels (say, $c_1 < c_2 < \dots < c_N$, although the exact meaning of these values is of no importance here).

The market is operating in continuous time $t \in \mathbb{R}_+$ where $\mathbb{R}_+ = [0, \infty)$. (As was mentioned above, the earlier version [1] used a more cumbersome discrete-time version of the underlying process.) At a given time $t \in \mathbb{R}_+$, there are $b_i(t)$ traders prepared to buy a unit of the commodity at price c_i and $s_i(t)$ traders prepared to sell it at this price, which leads to vectors

$$\mathbf{b}(t) = (b_1(t), \dots, b_N(t)), \quad \mathbf{s}(t) = (s_1(t), \dots, s_N(t)) \in \mathbb{Z}_+^N.$$

Here and below $\mathbb{Z}_+ = \{0, 1, \dots\}$ stands for a non-negative integer half-lattice and \mathbb{Z}_+^N for the non-negative integer N -dimensional lattice orthant. The pair $(\mathbf{b}(t), \mathbf{s}(t))$ represents a state of a Markov process $\{\mathbf{U}(t)\}$ that will be the subject of our analysis.

Suppose that, for given $k = 1, \dots, N$ and $t \in \mathbb{R}_+$, we have that $b_k(t) \geq s_k(t) > 0$ then each of the sellers gets a trade at a given rate $\rho_T > 0$ and leaves the market, together with one of the buyers. Therefore, both values $b_k(t)$ and $s_k(t)$ decrease at rate ρ_T . In addition, each one among $s_k(t)$ sellers (i) quits the market at rate $\rho_Q > 0$ or (ii) moves to the price level c_{k-1} at rate $\rho_M > 0$, if $k > 1$. Similarly, every buyer among the $b_k(t)$ buyers (i) quits the market at the same rate ρ_Q as above or (ii) moves to the price level c_{k+1} , again at rate ρ_M , provided that $k < N$.

Symmetrically, if $s_k(t) \geq b_k(t) > 0$ then each of the buyers gets a trade at rate ρ_T and leaves the market, together with his seller companion. The remaining traders at the price level c_k proceed in a manner as above.

Further, when $k = N$, a buyer leaves the system with rate $\rho_Q + \rho_M$. Similarly, for $k = 1$, a seller leaves the system with rate $\rho_Q + \rho_M$.

Finally, a random Poisson flow of new exogenous buyers arrives at the price level c_1 ; the rate of this arrival equals $\lambda_B > 0$. Similarly, a Poisson random flow of new sellers arrives at the price level c_N ; the rate of this arrival is $\lambda_S > 0$.

As usually, standard independence assumptions are in place. This generates the aforementioned Markov process $\{\mathbf{U}(t)\}$ with trajectories $\{(\mathbf{b}(t), \mathbf{s}(t))\}$, $t \in \mathbb{R}_+$.

Theorem 1 *For any values of parameters $\lambda_{B/S}$, $\rho_{Q/M}$ and ρ_T , the process $\{\mathbf{U}(t)\}$ is irreducible, aperiodic and positive recurrent. Therefore, it has a unique set of equilibrium probabilities $\pi = (\pi(\mathbf{b}, \mathbf{s}) : \mathbf{b}, \mathbf{s} \in \mathbb{Z}_+^N)$, and for any initial state $\mathbf{U}(0)$ (deterministic or random), the distribution of the random state $\mathbf{U}(t)$ at time t converges weakly to π as $t \rightarrow \infty$:*

$$\lim_{t \rightarrow \infty} \mathbb{P}(\mathbf{U}(t) = (\mathbf{b}, \mathbf{s})) = \pi(\mathbf{b}, \mathbf{s}).$$

Proof of Theorem 1. Irreducibility and aperiodicity of the process is evident. Positive recurrence follows from the following observation. The (random) time that a given trader (a buyer or a seller) spends in the system, i.e., the time from his arrival till exit, is majorized by a sum of N independent exponential variables. Therefore, the process $\{\mathbf{U}(t)\}$ can be majorized, in a natural fashion, by an $M/M/\infty$ queueing process. But the latter is known to be positive recurrent.

The remaining assertions of Theorem 1 are standard. \blacktriangle

Despite a concise description, the detailed pattern of behavior of process $\{\mathbf{U}(t)\}$ is rather complex, particularly for large values of N . For instance, consider the differences between vectors $\mathbf{b}(t')$ and $\mathbf{b}(t)$ and between $\mathbf{s}(t')$ and $\mathbf{s}(t)$, on a time interval (t, t') where $0 < t < t'$. The increments for the entries $b_k(\cdot)$ and $s_k(\cdot)$ for $1 \leq k \leq N-1$ are captured by the following equations:

$$b_k(t') = b_k(t) + i_{k-1}^M(t, t') - n_k(t, t') - i_k^M(t, t') - i_k^Q(t, t'),$$

and

$$s_k(t') = s_k(t) + j_{k+1}^M(t, t') - n_k(t, t') - j_k^M(t, t') - j_k^Q(t, t').$$

Here $i_k^M(t, t')$ is the number of buyers who move within time interval (t, t') from level k to $k+1$ and $j_k^M(t, t')$ that of sellers who move from level k to $k-1$. Next, $i_k^Q(t, t')$ is the number of buyers who quit the system during interval (t, t') from level k and $j_k^Q(t, t')$ the number of sellers who quit the system from level k . Finally, $n_k(t, t')$ is the number of buyers and sellers who got a trade over (t, t') at level k . All listed quantities are non-negative integer-values random variables. For $k=1$ the structure of the expression is similar, with the term $i_{k-1}^M(t, t')$ being replaced by $i_B(t, t') \geq 0$, while for $k=N$ the term $j_{k+1}^M(t, t')$ is replaced by $j_S(t, t') \geq 0$; both $i_B(t, t')$ and $j_S(t, t')$ being distributed according to a Poisson law with mean $\lambda_{B/S}(t' - t)$.

In the simplest case of a market with one price level ($N=1$), the process $\{\mathbf{U}(t)\}$ is a continuous-time random walk on the two-dimensional lattice quadrant \mathbb{Z}_+^2 , where

$$b(t') = b(t) + i_B(t, t') - n(t, t') - i_Q(t, t')$$

and

$$s(t') = s(t) + j_S(t, t') - n(t, t') - j_Q(t, t').$$

This already makes analytical representations for the invariant distribution π rather complicated; cf. [11] and references therein.

3 Scaling limit

The complexity of the time-dynamics and of the equilibrium distribution π for process $\{\mathbf{U}(t)\}$ makes it desirable to develop efficient methods of approximation. In this paper we focus on one such method based on scaling the parameters of the process (including states and time-steps).

The re-scaling procedure is as follows: we fix values $\gamma > 0$, $\beta > 0$, $\alpha > 0$, $\lambda_B > 0$ and $\lambda_S > 0$ and set:

$$\rho_T = \frac{\gamma}{L}, \quad \rho_Q = \frac{\beta}{L}, \quad \rho_M = \frac{\alpha}{L}. \quad (1)$$

In addition, we re-scale the states and the time: pictorially,

$$x_k \sim \frac{b_k}{L}, \quad y_k \sim \frac{s_k}{L}, \quad \tau \sim \frac{t}{L}. \quad (2)$$

Formally, denoting the Markov process generated for a given L by $\mathbf{U}^{(L)}$, we consider the continuous-time process

$$\mathbf{V}^{(L)}(\tau) = \frac{1}{L} \mathbf{U}^{(L)}(\tau L), \quad \tau \geq 0. \quad (3)$$

Let \mathbb{R}_+^N denote a positive orthant in N dimensions. Suppose we are given a pair of vectors $(\mathbf{x}(0), \mathbf{y}(0)) \in \mathbb{R}_+^N \times \mathbb{R}_+^N$ where $\mathbf{x}(0) = (x_1(0), \dots, x_N(0))$, $\mathbf{y}(0) = (y_1(0), \dots, y_N(0))$. Consider the following system of first-order ODEs for functions $x_k = x_k(\tau)$ and $y_k = y_k(\tau)$ where $\tau > 0$ and $1 \leq i \leq N$:

$$\begin{aligned} \dot{x}_1 &= \lambda_B - (\beta + \alpha)x_1 - \gamma \min[x_1, y_1], \\ \dot{x}_k &= \alpha x_{k-1} - (\beta + \alpha)x_k - \gamma \min[x_k, y_k], \quad 1 < k \leq N, \\ \dot{y}_k &= \alpha y_{k+1} - (\beta + \alpha)y_k - \gamma \min[x_k, y_k], \quad 1 \leq k < N, \\ \dot{y}_N &= \lambda_S - (\beta + \alpha)y_N - \gamma \min[x_N, y_N], \end{aligned} \quad (4)$$

with the initial data

$$x_k(0) \geq 0, \quad y_k(0) \geq 0, \quad 1 \leq k \leq N. \quad (5)$$

The fixed point $(\mathbf{x}^*, \mathbf{y}^*)$ of system (4) has

$$\mathbf{x}^* = (x_1^*, \dots, x_N^*) \quad \text{and} \quad \mathbf{y}^* = (y_1^*, \dots, y_N^*)$$

where x_k^* and y_k^* give a solution to

$$\begin{aligned} \lambda_B &= (\beta + \alpha)x_1^* + \gamma \min[x_1^*, y_1^*], & (1') \\ \alpha x_{k-1}^* &= (\beta + \alpha)x_k^* + \gamma \min[x_k^*, y_k^*], \quad 1 < k \leq N, & (2') \\ \alpha y_{k+1}^* &= (\beta + \alpha)y_k^* + \gamma \min[x_k^*, y_k^*], \quad 1 \leq k < N, & (3') \\ \lambda_S &= (\beta + \alpha)y_N^* + \gamma \min[x_N^*, y_N^*]. & (4') \end{aligned} \quad (6)$$

(In (6) we noted individual equations by addition signs that are used below). Both systems (4) and (6) are non-linear. However, the non-linearity ‘disappears’ at a local level which greatly simplifies the analysis of these systems.

In Theorems 2 and 3 below, we use the distance generated by the Euclidean norm in $\mathbb{R}^N \times \mathbb{R}^N$.

Theorem 2 (a) For any initial data $(\mathbf{x}(0), \mathbf{y}(0))$ with $x_i(0) \geq 0$, $y_i(0) \geq 0$, $1 \leq i \leq N$ for all $\tau > 0$ there exists a unique solution $(\mathbf{x}(\tau), \mathbf{y}(\tau))$, to problem (4)–(5) and $x_i(\tau) > 0$, $y_i(\tau) > 0$.

(b) As $\tau \rightarrow \infty$, the solution approaches a fixed point which is a unique solution to system (6):

$$\text{dist}[(\mathbf{x}(\tau), \mathbf{y}(\tau)), (\mathbf{x}^*, \mathbf{y}^*)] \rightarrow 0. \quad (7)$$

Proof of Theorem 2, (a). Obviously, a unique solution exists for sufficiently small $\tau > 0$. Suppose that $x_i = \max_k x_k$ and $\max_{k < i} x_k < x_i$. Then $\dot{x}_i \leq 0$. Similarly if $y_j = \max_k y_k$ and $\max_{k > j} y_k < y_j$ then $\dot{y}_j \leq 0$. Also, $\dot{x}_1 < 0$ if $x_1 > \lambda_B/(\beta + \alpha)$ and $\dot{y}_N < 0$ if $y_N > \lambda_S/(\beta + \alpha)$. Moreover, $\dot{x}_1 \geq 0$ when $x_1 = 0$ and $\dot{y}_N \geq 0$ when $y_N = 0$. Thus the components of the solution $x_i(\tau)$ and $y_i(\tau)$ are non-negative and uniformly bounded. By standard constructions of the ODE theory, a unique solution $\{(\mathbf{x}(\tau), \mathbf{y}(\tau))\}$ exists for all $\tau > 0$, and $\mathbf{x}(\tau), \mathbf{y}(\tau) \in \mathbb{R}_+^N$. \blacktriangle

Before proving assertion (b), we discuss several properties of the solution to (4)–(5). The following Proposition 1 indicates that the solution to (4) possesses a kind of the min/max principle.

Proposition 1 Suppose that we have two initial points, $\mathbf{x} = (x_1, \dots, x_n)$ and $\mathbf{x}' = (x'_1, \dots, x'_n) \in \mathbb{R}_+^N$ such that for some $k = 1, \dots, N$,

$$x_k \leq x'_k, \text{ and } y_k \geq y'_k. \quad (8)$$

Then for the solutions $(\mathbf{x}(\tau), \mathbf{y}(\tau))$, $(\mathbf{x}'(\tau), \mathbf{y}'(\tau))$, to (4), with the initial conditions $(\mathbf{x}(0), \mathbf{y}(0)) = (\mathbf{x}, \mathbf{y})$ and $(\mathbf{x}'(0), \mathbf{y}'(0)) = (\mathbf{x}', \mathbf{y}')$, for all $\tau > 0$,

$$x_k(\tau) \leq x'_k(\tau), \text{ and } y_k(\tau) \geq y'_k(\tau). \quad (9)$$

Similarly if $x_k \geq x'_k$ and $y_k \leq y'_k$ then the corresponding solutions obey $x_k(\tau) \geq x'_k(\tau)$ and $y_k(\tau) \leq y'_k(\tau)$, for all $\tau > 0$

Proof of Proposition 1. It is sufficient to consider the situation where the strict inequalities take place. Suppose that (9) holds strictly for $\tau < \tau_0$ and fails at $\tau = \tau_0$. For instance, let $x_k(\tau_0) = x'_k(\tau_0)$ and assume that $k > 1$ is the minimal index for which such an equality takes place.

If $y_k(\tau_0) > y'_k(\tau_0)$ then $\dot{x}_k(\tau_0) < \dot{x}'_k(\tau_0)$ and the above equality $x_k(\tau_0) = x'_k(\tau_0)$ is impossible. The other case is considered in a similar manner. \blacktriangle

A corollary of proposition 1 is

Proposition 2 If $\dot{x}_k(0) \geq 0$ and $\dot{y}_k(0) \leq 0$ for all $1 \leq k \leq N$ then $x_k(\tau)$ increases and $y_k(\tau)$ decreases in τ , for all $\tau > 0$ and $1 \leq k \leq N$. Similarly if $\dot{x}_k(0) \leq 0$ and $\dot{y}_k(0) \geq 0$, $1 \leq k \leq N$, then $x_k(\tau)$ decreases and $y_k(\tau)$ increases in τ .

Proof of Proposition 2. It again suffices to assume that the strict inequalities hold true: $\dot{x}_k(0) > 0$ and $\dot{y}_k(0) < 0$. Then, for a small $\delta > 0$: $x_k(\delta) > x_k(0)$ and $y_k(\delta) < y_k(0)$.

Set $x'_k(0) = x_k(\delta)$, $y'_k(0) = y_k(\delta)$. The coefficients of equations do not depend on τ , therefore $x'_k(\tau) = x_k(\tau + \delta)$ and $y'_k(\tau) = y_k(\tau + \delta)$ for all $\tau > 0$. By Proposition 1, $x_k(\tau + \delta) = x'_k(\tau) \geq x_k(\tau)$ and $y'_k(\tau + \delta) = y'_k(\tau) \leq y_k(\tau)$, for all $\tau > 0$ and $1 \leq k \leq N$. As δ may be arbitrarily small, the assertion of Proposition 2 is valid. \blacktriangle

Proof of Theorem 2, (b). Given $\mathbf{x}(0) = (x_1(0), \dots, x_N(0))$ and $\mathbf{y}(0) = (y_1(0), \dots, y_N(0)) \in \mathbb{R}_+^N$, let $(\mathbf{x}(\tau), \mathbf{x}'(\tau))$ be the solution to (4), (5). Consider two additional solutions, $(\mathbf{x}'(\tau), \mathbf{y}'(\tau))$ and $(\mathbf{x}''(\tau), \mathbf{y}''(\tau))$, to (4) with

$$x'_k(0) = 0, \quad y'_k(0) = \max \left[\lambda_S / (\alpha + \beta), \max_i y_i \right],$$

and

$$x''_k(0) = \max \left[\lambda_B / (\alpha + \beta), \max_i x_i(0) \right], \quad y''_k(0) = 0.$$

By Propositions 1 and 2

$$x'_k(\tau) \leq x_k(\tau) \leq x''_k(\tau), \quad y'_k(\tau) \geq y_k(\tau) \geq y''_k(\tau). \quad (10)$$

Further, $x'_k(\tau)$ increases, while $y'_k(\tau)$ decreases in τ . By the same token, $x''_k(\tau)$ decreases and $y''_k(\tau)$ increases in τ . Therefore, both pairs $(\mathbf{x}'(\tau), \mathbf{y}'(\tau))$ and $(\mathbf{x}''(\tau), \mathbf{y}''(\tau))$ tend to limits as $\tau \rightarrow \infty$, which are fixed points, i.e., solutions to Eqns (6). By (10), any solution to (4), (5) eventually lies between these limits. To finish the proof of the theorem, we have to show that the solution $(\mathbf{x}^*, \mathbf{y}^*)$ to Eqn (6) is unique.

For convenience, we state the corresponding assertion as Lemma 1.

Lemma 1 *For any values $\lambda_{B/S}, \alpha, \gamma$ and $\beta \geq 0$ there exists a unique solution to Eqns (6).*

Proof of Lemma 1. To start with, note that every solution to (6) has

$$x_1^* > x_2^* > \dots > x_N^*, \quad y_1^* < y_2^* < \dots < y_N^*.$$

Therefore, if $x_1 \leq y_1$ then $x_k < y_k$, $1 < k \leq N$ and if $x_N \geq y_N$ then $x_k > y_k$, $1 \leq k < N$.

It is convenient to introduce auxiliary variables $v_k, w_k \geq 0$ in terms of which Eqns (6) will be treated. Geometrically the idea is as follows: we start with Eqn (6)⁽¹⁾: $\lambda_B = (\alpha + \beta)v_1 + \min[v_1, w_1]$ and watch how this relation between v_1, w_1 is transformed by (6)⁽²⁾, (6)⁽³⁾ to relation between v_2, w_2 , then to relation between v_3, w_3 etc.

The locus of points $(u, v) \in \mathbb{R}^2$ where

$$v, w > 0, \quad \lambda_B = (\alpha + \beta)v + \min[v, w]$$

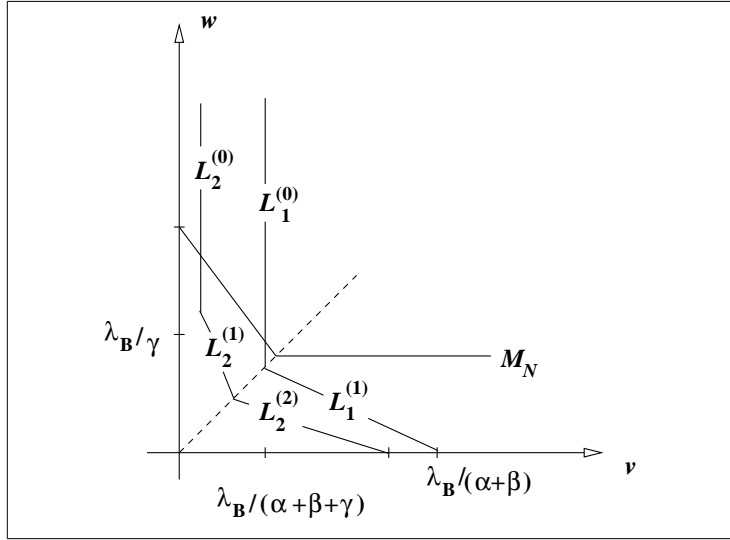
coincides with a continuous broken line, $L_1 \subset \mathbb{R}_+^2$, formed by two pieces: 1) a vertical ray $L_1^{(0)}$ emitted from the point $o_1^{(0)}$ lying on the bisectrix where

$$o_1^{(0)} = (\lambda_B/(\alpha + \beta + \gamma), \lambda_B/(\alpha + \beta + \gamma))$$

and 2) a line segment $L_1^{(1)}$ of a negative slope $-(\alpha + \beta)/\gamma$, joining $o_1^{(0)}$ with the point $o_1^{(1)}$ on the horizontal axis:

$$o_1^{(1)} = (\lambda_B/(\alpha + \beta), 0).$$

See the figure below.



The passage from values v_1, w_1 to v_2, w_2 generates a $1 - 1$ map acting on L_1 . The image of L_1 under the map $(v_1, w_1) \mapsto (v_2, w_2)$ is another continuous broken line, $L_2 \subset \mathbb{R}_+^2$, formed by three pieces: 1) a vertical ray $L_2^{(0)}$ issued from point

$$o_2^{(0)} = (\lambda_B \alpha / (\alpha + \beta + \gamma)^2, \lambda_B / \alpha),$$

2) a line segment $L_2^{(1)}$ joining the points $o_2^{(0)}$ and $o_2^{(1)}$ where $o_2^{(1)}$ lies on the bisectrix and has both co-ordinates equal to

$$\frac{\lambda_B \alpha (\alpha + \beta + \gamma)}{(\alpha + \beta + \gamma) [(\alpha + \beta)(\alpha + \beta + \gamma)^2 + \alpha^2 \gamma]},$$

and 3) a line segment $L_2^{(2)}$ joining $o_2^{(1)}$ with the point $o_2^{(2)}$ on the horizontal axis:

$$o_2^{(2)} = \left(\frac{\lambda_B \alpha}{(\alpha + \beta)^2}, 0 \right).$$

The slopes dw/dv of segments $L_2^{(1)}$ and $L_2^{(2)}$ are negative, but the slope flattens when we pass from $L_2^{(1)}$ to $L_2^{(2)}$.

In the above figure, L_1 and L_2 are shown on the same (v, w) -plane \mathbb{R}^2 ; in this figure line L_2 lies to the left of L_1 . (The third broken line, M_N present in the figure is explained below.)

A similar picture persists when we iterate, i.e., pass from (v_2, w_2) to (v_3, w_3) and so on. At step k $(v_k, w_k) \mapsto (v_{k+1}, w_{k+1})$ where again the map is 1 - 1, (v_{k+1}, w_{k+1}) belongs to a continuous broken line L_{k+1} in \mathbb{R}_+^2 formed by $k + 2$ pieces. One piece, $L_{k+1}^{(0)}$, is a vertical ray while the $k + 1$ others, $L_{k+1}^{(1)}, \dots, L_{k+1}^{(k+1)}$, are line segments of negative slopes, the slope flattens when we pass from $L_k^{(1)}$ to $L_k^{(k)}$. The last segment, $L_{k+1}^{(k+1)}$, joins a point on the bisectrix and a point on the horizontal axis.

At the end of this process we obtain a continuous broken line L_N , the locus of points (v_N, w_N) . Our next step is to consider the intersection of L_N and M_N where M_N is the locus where

$$\lambda_S = (\alpha + \beta)w_N + \gamma \min [v_N, w_N].$$

More precisely, M_N is a continuous broken line formed by a horizontal ray issued from the point

$$\left(\frac{\lambda_S}{\alpha + \beta + \gamma}, \frac{\lambda_S}{\alpha + \beta + \gamma} \right)$$

and a line segment joining this point with the point

$$\left(0, \frac{\lambda_S}{\alpha + \beta} \right)$$

lying on the vertical axis. Cf. the figure.

We want to check that the point of intersection is always unique: it yields to unique solution to (6).

Line L_N may intersect the horizontal part of M_N . That means that there exist a solution to (6) where $x_1^* > y_1^*$ and $x_N^* > y_N^*$ (this case is not presented on our figure). For the proof of lemma it is needed to show that in this case L_N cannot intersect the sloppy part of M_N , where $y_N > x_N$. For that it is sufficient to prove that dw/dv on $L_N \setminus L_N^{(N)}$, the part of L_N above the bisectrix, is steeper than $-\gamma/(\alpha + \beta)$, the slope of the segment of line M_N .

On the other hand if L_N does not intersect the horizontal part of M_N it has to intersect the sloppy part of M_N and it is needed to show that in this case such intersection is unique. Here again it is sufficient to show that dw/dv on $L_N \setminus L_N^{(N)}$ is steeper than $-\gamma/(\alpha + \beta)$.

To prove the assertion of the lemma we show that dw/dv on $L_N \setminus L_N^{(N)}$ is always steeper than $-\gamma/(\alpha + \beta)$. In fact, it suffices to verify that

$$\frac{dw_N}{dv_N} < -\frac{\gamma}{\alpha + \beta} \quad \text{on } L_N^i, \quad 1 \leq i < N. \quad (11)$$

Any segment $L_k^{(i)}$, $1 < i < k$, maps onto segment $L_{k+1}^{(i)}$, segment $L_k^{(k)}$ maps onto two segments: $L_{k+1}^{(k)}$ and $L_{k+1}^{(k+1)}$. The slope of segment $L_{k+1}^{(i)}$, $i = 1, \dots, k+1$ is steeper than that of $L_k^{(i)}$ and the slope of $L_{k+1}^{(k+1)}$ is steeper than that of $L_k^{(k)}$. More, the slopes of $L_N^{(i)}$, $0 < i \leq N$, are steeper then the steep segment of M_N . To show that consider three cases of the map $L_k \rightarrow L_{k+1}$:

- 1) $v_k > w_k$, $v_{k+1} > w_{k+1}$;
- 2) $v_k > w_k$, $v_{k+1} < w_{k+1}$;
- 3) $v_k < w_{k+1}$, $v_{k+1} < w_{k+1}$.

In these cases we have, the following equations

$$\frac{dw_{k+1}}{dv_{k+1}} = \begin{cases} \frac{A_{k+1}(\alpha + \beta + \gamma)}{\alpha^2} \frac{dw_k}{dv_k}, \\ \frac{(\alpha + \beta + \gamma)^2}{\alpha^2} \frac{dw_k}{dv_k}, \\ \frac{B_k(\alpha + \beta + \gamma)}{\alpha^2} \frac{dw_k}{dv_k}, \end{cases} \quad (12)$$

where, respectively,

$$\begin{cases} v_k = \frac{(\alpha + \beta)v_{k+1} + \gamma w_{k+1}}{\alpha}, & w_{k+1} = \frac{(\alpha + \beta + \gamma)w_k}{\alpha}, \\ v_k = \frac{(\alpha + \beta + \gamma)v_{k+1}}{\alpha}, & w_{k+1} = \frac{(\alpha + \beta + \gamma)w_k}{\alpha}, \\ v_k = \frac{(\alpha + \beta + \gamma)v_{k+1}}{\alpha}, & w_{k+1} = \frac{(\alpha + \beta)w_k + \gamma v_k}{\alpha}. \end{cases}$$

Here $A_{k+1} = \alpha + \beta + \gamma \frac{w_{k+1}}{v_{k+1}}$ and $B_k = \alpha + \beta + \gamma \frac{v_k}{w_k}$.

Therefore for $N > 2$

$$\begin{aligned} \left| \frac{dw_N}{dv_N} \right| &> \left| \frac{(\alpha + \beta + \gamma)^{N-1}}{\alpha^{N-1}} \frac{dw_1}{dv_1} \right| = \frac{(\alpha + \beta + \gamma)^{N-1}}{\alpha^{N-1}} \frac{\alpha + \beta}{\gamma} \\ &> \left| \frac{dw}{dv} \right|_{M_N} = \frac{\gamma}{\alpha + \beta}. \end{aligned}$$

For $N = 2$ we have the middle case in (12) with $k = 1$, $k + 1 = 2$. Here again the needed inequality takes place. In fact, $\frac{dw_1}{dv_1} = -\frac{\alpha + \beta}{\gamma}$. By using (12), we get that for $N = 2$,

$$\frac{dw_2}{dv_2} = -\frac{(\alpha + \beta + \gamma)^2}{\alpha^2} \frac{dw_1}{dv_1} \text{ on } L_2^1.$$

This finishes the proof of Lemma 1. \blacktriangle

Theorem 3 Suppose that the re-scaled initial states converge in probability: for any $\epsilon > 0$,

$$\lim_{L \rightarrow \infty} \mathbb{P} \left(\text{dist} \left[\frac{1}{L} \mathbf{U}(0), (\mathbf{x}(0), \mathbf{y}(0)) \right] \geq \epsilon \right) = 0.$$

Then, for all $T > 0$, the process $\left\{ \mathbf{V}^{(L)}(\tau), \tau \in [0, T] \right\}$ converges in probability to the solution $\left\{ (\mathbf{x}(\tau), \mathbf{y}(\tau)), 0 \leq \tau \leq T \right\}$. That is, $\forall \epsilon > 0$,

$$\lim_{L \rightarrow \infty} \mathbb{P} \left(\sup_{0 \leq \tau \leq T} \left\{ \text{dist} \left[\mathbf{V}^{(L)}(\tau), (\mathbf{x}(\tau), \mathbf{y}(\tau)) \right] \right\} \geq \epsilon \right) = 0. \quad (13)$$

In particular, if $\mathbf{x}(0) = \mathbf{x}^*$ and $\mathbf{y}(0) = \mathbf{y}^*$ then

$$\lim_{L \rightarrow \infty} \mathbb{P} \left(\sup_{0 \leq \tau \leq T} \left\{ \text{dist} \left[\mathbf{V}^{(L)}(\tau), (\mathbf{x}^*, \mathbf{y}^*) \right] \right\} \geq \epsilon \right) = 0. \quad (14)$$

Moreover, if process $\{\mathbf{U}(t), \}$ is in equilibrium then Eqn (14) holds true.

Proof of Theorem 3. Let $\mathbf{G}^{(L)}$ denote the generator of the Markov process $\{\mathbf{U}^{(L)}(t)\}$ (with rates as in Eqn (1)). Then the action of matrix $\mathbf{G}^{(L)}$ on functions $\phi(\mathbf{b}, \mathbf{s})$ of state variables $\mathbf{b}, \mathbf{s} \in \mathbb{Z}_+^N$ is determined by the equation

$$\begin{aligned} \mathbf{G}^{(L)} \phi(\mathbf{b}, \mathbf{s}) = & \frac{\beta}{L} \sum_{1 \leq k \leq N} \left([\phi(\mathbf{b} - \mathbf{e}_k, \mathbf{s}) - \phi(\mathbf{b}, \mathbf{s})] b_k \right. \\ & \left. + [\phi(\mathbf{b}, \mathbf{s} - \mathbf{e}_k) - \phi(\mathbf{b}, \mathbf{s})] s_k \right) \\ & + \frac{\alpha}{L} \left(\sum_{1 \leq k < N} [\phi(\mathbf{b} - \mathbf{e}_k + \mathbf{e}_{k+1}, \mathbf{s}) - \phi(\mathbf{b}, \mathbf{s})] b_k \right. \\ & \left. + \sum_{1 < k \leq N} [\phi(\mathbf{b}, \mathbf{s} - \mathbf{e}_k + \mathbf{e}_{k-1}) - \phi(\mathbf{b}, \mathbf{s})] s_k \right) \\ & + \lambda_B [\phi(\mathbf{b} + \mathbf{e}_1, \mathbf{s}) - \phi(\mathbf{b}, \mathbf{s})] + \lambda_S [\phi(\mathbf{b}, \mathbf{s} + \mathbf{e}_N) - \phi(\mathbf{b}, \mathbf{s})] \\ & + \frac{\gamma}{L} \sum_{1 \leq k \leq N} [\phi(\mathbf{b} - \mathbf{e}_k, \mathbf{s} - \mathbf{e}_k) - \phi(\mathbf{b}, \mathbf{s})] (b_k \wedge s_k). \end{aligned} \quad (15)$$

Here \mathbf{e}_k , $1 \leq k \leq N$, stands for the vector in \mathbb{Z}_+^N whose components are all 0's except for the k th one, equal to 1.

In particular, we can take a function $\phi = \phi^{(L)}$ of the form $\phi(\mathbf{b}, \mathbf{s}) = \varphi\left(\frac{\mathbf{b}}{L}, \frac{\mathbf{s}}{L}\right)$ where φ is a smooth function on \mathbb{R}_+^N . This choice agrees with the spatial scaling $\mathbf{x} \sim \mathbf{b}/L$, $\mathbf{y} \sim \mathbf{s}/L$ in Eqn (2). Then $\forall \mathbf{x} = (x_1, \dots, x_N) \in \mathbb{R}_+^N$ and $\mathbf{y} = (y_1, \dots, y_N) \in \mathbb{R}_+^N$, with $\mathbf{b} = \lfloor L\mathbf{x} \rfloor$, $\mathbf{s} = \lfloor L\mathbf{y} \rfloor$ where $\lfloor \cdot \rfloor$ stands for the

integer part, we obtain that

$$\begin{aligned}
& \mathbf{G}^{(L)} \varphi(\lfloor L\mathbf{x} \rfloor / L, \lfloor L\mathbf{y} \rfloor / L) \\
&= \beta \sum_{1 \leq k \leq N} \left\{ \left[\varphi\left(\frac{\mathbf{b} - \mathbf{e}_k}{L}, \frac{\mathbf{s}}{L}\right) - \varphi\left(\frac{\mathbf{b}}{L}, \frac{\mathbf{s}}{L}\right) \right] b_k / L \right. \\
&\quad \left. + \left[\varphi\left(\frac{\mathbf{b}}{L}, \frac{\mathbf{s} - \mathbf{e}_k}{L}\right) - \varphi\left(\frac{\mathbf{b}}{L}, \frac{\mathbf{s}}{L}\right) \right] s_k / L \right\} \\
&+ \alpha \left\{ \sum_{1 \leq k < N} \left[\varphi\left(\frac{\mathbf{b} - \mathbf{e}_k + \mathbf{e}_{k+1}}{L}, \frac{\mathbf{s}}{L}\right) - \varphi\left(\frac{\mathbf{b}}{L}, \frac{\mathbf{s}}{L}\right) \right] b_k / L \right. \\
&\quad \left. + \sum_{1 < k \leq N} \left[\varphi\left(\frac{\mathbf{b}}{L}, \frac{\mathbf{s} - \mathbf{e}_k + \mathbf{e}_{k-1}}{L}\right) - \varphi\left(\frac{\mathbf{b}}{L}, \frac{\mathbf{s}}{L}\right) \right] s_k / L \right\} \\
&\quad + \lambda_B \left[\varphi\left(\frac{\mathbf{b} + \mathbf{e}_1}{L}, \frac{\mathbf{s}}{L}\right) - \varphi\left(\frac{\mathbf{b}}{L}, \frac{\mathbf{s}}{L}\right) \right] \\
&\quad + \lambda_S \left[\varphi\left(\frac{\mathbf{b}}{L}, \frac{\mathbf{s} + \mathbf{e}_N}{L}\right) - \varphi\left(\frac{\mathbf{b}}{L}, \frac{\mathbf{s}}{L}\right) \right] \\
&+ \gamma \sum_{1 \leq k \leq N} \left[\varphi\left(\frac{\mathbf{b} - \mathbf{e}_k}{L}, \frac{\mathbf{s} - \mathbf{e}_k}{L}\right) - \varphi\left(\frac{\mathbf{b}}{L}, \frac{\mathbf{s}}{L}\right) \right] (b_k \wedge s_k) / L.
\end{aligned}$$

Next, we multiply the both side by L – in agreement with the time-scale $\tau \sim t/L$ in Eqn (2) – and pass to the limit $L \rightarrow \infty$. This yields

$$\begin{aligned}
& \lim_{L \rightarrow \infty} L \mathbf{G}^{(L)} \varphi(\mathbf{x}, \mathbf{y}) \\
&= \left\{ -\beta \sum_{1 \leq k \leq N} \left(x_k \frac{\partial}{\partial x_k} + y_k \frac{\partial}{\partial y_k} \right) \right. \\
&\quad + \alpha \left[\sum_{1 \leq k < N} x_k \left(\frac{\partial}{\partial x_{k+1}} - \frac{\partial}{\partial x_k} \right) + \sum_{1 < k \leq N} y_k \left(\frac{\partial}{\partial y_{k-1}} - \frac{\partial}{\partial y_k} \right) \right] \\
&\quad \left. + \left(\lambda_B \frac{\partial}{\partial x_1} + \lambda_S \frac{\partial}{\partial y_N} \right) - \gamma \sum_{1 \leq k \leq N} (x_k \wedge y_k) \left(\frac{\partial}{\partial x_i} + \frac{\partial}{\partial y_i} \right) \right\} \varphi(\mathbf{x}, \mathbf{y}).
\end{aligned}$$

Applying Theorem 6.1 from [12], we obtain Eqn (13).

The next remark is that each scaled process $\{\mathbf{V}^{(L)}\}$ has a unique invariant distribution $\hat{\pi}^{(L)}$; the family of probability distributions $\hat{\pi}^{(L)}$ (considered on \mathbb{R}_+^N) is compact in the sense of convergence in probability. This can be deduced from the above remark that the original processes $\{\mathbf{U}^{(L)}\}$, and hence, the scaled process $\{\mathbf{V}^{(L)}\}$ can be majorized by suitable analogs of M/M/ ∞ queueing systems. It is easy to see that every limiting point for $\hat{\pi}^{(L)}$ when $L \rightarrow \infty$ is a delta-measure sitting at a fixed point for system (5). However, the latter is unique and coincides with $(\mathbf{x}^*, \mathbf{y}^*)$. Consequently, the distributions $\hat{\pi}^{(L)}$ converge in probability to the aforementioned delta-measure. Then, applying the already established result, we obtain Eqn (14). \blacktriangle

4 Fixed points in the scaling limit. Concluding remarks

The approximation developed in Theorem 3 calls for an analysis of solutions to (6).

The parameter space \mathbb{R}_+^5 formed by γ , $\alpha_{Q/M}$, and $\lambda_{b/s}$ is partitioned into open domains where one of the following generic patterns persists:

(i) $v_N > w_N$, (ii) $v_1 < w_1$, and (iii) $v_i > w_i$ for $i = 1, \dots, \ell$ and $v_i < w_i$ for $i = \ell + 1, \dots, N$ where $1 < \ell < N$. In each of these domains system (6) is linear.

A particular algorithm for calculating $(\mathbf{x}^*, \mathbf{y}^*)$ is based on the following recursion. Set

$$x_1^{(0)} = \frac{\lambda_b}{\alpha_Q + \alpha_M + \gamma}, \quad x_i^{(0)} = \frac{\alpha_M x_{i-1}^{(0)}}{\alpha_Q + \alpha_M + \gamma}, \quad y_N^{(0)} = \frac{\lambda_s}{\alpha_Q + \alpha_M}, \quad y_i^{(0)} = \frac{\alpha_M y_{i+1}^{(0)}}{\alpha_Q + \alpha_M}$$

Next, let $(\mathbf{x}^{(k)}, \mathbf{y}^{(k)})$, $k = 1, 2, \dots$ be the solution to the system

$$\begin{aligned} \lambda_b &= \left(\alpha_q + \alpha_m \right) x_1^{(k)} + \gamma \min [x_1^{(k-1)}, y_1^{(k-1)}], \\ \alpha_m x_{i-1}^{(k)} &= \left(\alpha_q + \alpha_m \right) x_i^{(k)} + \gamma \min [x_i^{(k-1)}, y_i^{(k-1)}], \quad 1 < i \leq N, \\ \alpha_m y_{i+1}^{(k)} &= \left(\alpha_q + \alpha_m \right) y_i^{(k)} + \gamma \min [x_i^{(k)}, y_i^{(k-1)}], \quad 1 \leq i < N, \\ \lambda_s &= \left(\alpha_q + \alpha_m \right) y_N^{(k)} + \gamma \min [x_N^{(k)}, y_N^{(k-1)}]. \end{aligned}$$

These iterations converge because the inequalities $x_i^{(k)} > x_i^{(k-1)}$, $y_i^{(k)} < y_i^{(k-1)}$ hold true $\forall i, k \geq 1$ and, values $x_i^{(k)}, y_i^{(k)}$ are uniformly bounded and there exist $\lim_{k \rightarrow \infty} x_i^{(k)}, \lim_{k \rightarrow \infty} y_i^{(k)}$ that, naturally, satisfy (6).

We conclude with the following remarks.

Our model presents also a caricature of "overproduction crisis": In fact, if λ_s is sufficiently large, so that $x_i < y_i$, $1 \leq i \leq N$, then the amount of "trades" $\sum_i \gamma \min[x_i, y_i] = \sum_i \gamma x_i$ is not changing by increase of λ_s , all "extra" sellers leave the market without performing any trade.

It is interesting to investigate the dependance of trade performance on γ as $\gamma \rightarrow \infty$, that is where the trade action happens almost immediately after the moment when traders appear at some price level. Then almost all trades happen at two levels i_0 and $i_0 + 1$, and x_i is very small as $i > i_0 + 1$, y_i is very small as $i < i_0$. But our limit model does not permit to consider the case $\gamma = \infty$, though, sure the initial Markov process can be investigated in case of immediate trade deals. The limiting case $\gamma = \infty$ of our model is close to the problems investigated in [10].

We hope that the variation of these models parameters can help to determine factors attracting or repelling various 'market participants'. An important aspect of any model of the market is what possibilities it gives for an accurate

prediction of the stochastic component in the dynamics of the market prices and volumes.

The current set-up of the model presented here admits straightforward generalizations to the case where parameters γ and $\alpha_{Q/M}$ depend on i , $0 < i < N$ and on the trader type (b/s). Another generalization emerges if these parameters and $\lambda_{b/s}$ become state-dependent. It is also possible to allow the exogenous buyers and sellers to enter the system at any price level among c_1, \dots, c_N . To take into account elements of the FCFS discipline, one could introduce various priorities into the dynamics of process $U(t)$.

Finally, we would like to note that there are several forms of convergence for which the assertion in Theorem 3 holds true.

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